Chapter 7 Actual & Conjectured results on the zeros of $\zeta(s)$

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The Functional Equation

Recall that we have the Riemann zeta function defined by

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}du \tag{1}$$

for $\operatorname{Re} s > 0$. It can be shown that for all $s \in \mathbb{C}$ it satisfies

Theorem 7.1 ***Functional Equation for the Riemann zeta function***

$$\zeta(s) = \frac{(2\pi)^s}{\pi}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\,. \tag{2}$$

Here

Definition 7.2 *The **Gamma function** is defined by*

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

for $\operatorname{Re} s > 0$.

Properties 1. $\Gamma(s)$ is **holomorphic** in $\operatorname{Re} s > 0$.

The following explanation is not examinable. The integral converges at $t = \infty$ for all $s \in \mathbb{C}$ because of the e^{-t} factor but converges at $t = 0$ only for $\operatorname{Re} s > 0$. Given any $\delta > 0$ we have

$$\left|e^{-t}t^{s-1}\right| \le e^{-t}t^{\delta-1},$$

and since

$$\int_0^\infty e^{-t}t^{\delta-1}dt < \infty$$

we have that the integral defining $\Gamma(s)$ converges *uniformly* for all $\operatorname{Re} s \ge \delta$. Weierstrass's Theorem for integrals can be shown to apply here, in which case the holomorphic properties of the integrand as a function of s transfer to $\Gamma(s)$, in particular $\Gamma(s)$ is holomorphic in $\operatorname{Re} s \ge \delta$. True for all $\delta > 0$ means that $\Gamma(s)$ is holomorphic in $\operatorname{Re} s > 0$.

2. $\Gamma(s)$ satisfies a **Functional Equation**,

$$\Gamma(s+1) = s\Gamma(s)\,, \tag{3}$$

which follows on integration by parts.

3. **Analytic Continuation**. Writing the functional equation as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s},$$

we see that the right hand side is holomorphic in $\mathrm{Re}\,(s+1) > 0$, i.e. $\mathrm{Re}\,s > -1$ except for a simple pole at $s = 0$. Thus we have an analytic continuation of $\Gamma(s)$ to $\mathrm{Re}\,s > -1$. This can be repeated, i.e.

$$\Gamma(s) = \frac{\Gamma(s+2)}{s\,(s+1)},$$

holomorphic for $\mathrm{Re}\,s > -2$ except for simply poles at $s = 0$ and $s = -1$. Continue, concluding that $\Gamma\,(s)$ has an analytic continuation to all of $\mathbb{C}$ with simple poles at $s = 0, -1, -2, -3, ...$.

4. **Important** It can be shown that the gamma function is **never zero**.

Note A particular case of the functional equation is when $s = n \in \mathbb{N}$, for then repeated applications of (3) gives

$$\Gamma(n+1) = n\Gamma(n) = n\,(n-1)\,(n-2)\,....1\Gamma(1)\,.$$

But

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1$$

so $\Gamma(n+1) = n!$ Thus the gamma function generalises the factorial function.

Deductions from the Functional Equation

• **Definition of** $\zeta\,(s)$ **for** $\mathrm{Re}\,s < 1$**.**

If $\mathrm{Re}\,s < 1$ then $\mathrm{Re}\,(s-1) > 0$ and the function

$$F(s) = \frac{(2\pi)^s}{\pi}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)\,,$$

is, using (1), well-defined. But what is F?

The functional equation, (2) , says that $F(s) = \zeta(s)$ where both $F\,(s)$ and $\zeta(s)$ are defined, i.e. $0 < \mathrm{Re}\,s < 1$. Hence F is the analyic continuation of ζ from $0 < \mathrm{Re}\,s < 1$ to $\mathrm{Re}\,s < 1$. Along with (1) this means that we have ζ defined at all points of $\mathbb{C}$.

• **Zeros** For $\mathrm{Re}\, s < 0$ we have $\mathrm{Re}\,(1-s) > 1$ and we know that the Riemann zeta function $\zeta(1-s)$ on the right hand side of (2) has no zeros for such s. We are told above that the Gamma function has no zeros. Thus any zeros of $\zeta\,(s)$ seen on the left hand side of (2) for $\mathrm{Re}\, s < 0$ arise from the zeros of $\sin(\pi s/2)$ which occur at the even integers. These zeros of $\zeta(s)$ at $-2, -4, -6, ..., -2m, ...$ are called the *trivial zeros* of $\zeta(s)$. Any other zeros ρ can only lie in the *critical strip* $0 \leq \mathrm{Re}\, \rho \leq 1$ and are called *critical zeros*. These critical zeros are normally denoted by $\rho = \beta + i\gamma$.

From the Functional Equation we see that if $\rho = \beta + i\gamma$ is a non-trivial zero then $1-\rho = 1-\beta-i\gamma$ is also a zero. But further

$$0 = \zeta\,(1-\rho) = \zeta\,(1-\beta-i\gamma) = \overline{\zeta\,(1-\beta+i\gamma)},$$

so $1-\beta+i\gamma$ is also a zero. Thus the non-trivial zeros are symmetric about both the horizontal line $\mathrm{Im}\, s = 0$ and the vertical line $\mathrm{Re}\, s = 1/2$.

Conjecture 7.3 ***Riemann Hypothesis*** *There are no critical zeros in the region* $\mathrm{Re}\, s > 1/2$.

The symmetry around the line $\mathrm{Re}\, s = 1/2$ means that the Riemann Hypothesis is equivalent to claiming that all critical zeros satisfy $\mathrm{Re}\, \rho = 1/2$.

• **Poles** We know from (1) that $\zeta(s)$ as only one pole in $\mathrm{Re}\, s > 0$, at $s = 1$. Yet from the functional equation it may appear that $\zeta(s)$ has poles when $\Gamma(1-s)$ has poles. But these are at $1-s = 0, -1, -2, -3, ...$, i.e. $s = 1, 2, 3, ...$ so we get no new poles in $\mathrm{Re}\, s \leq 0$.

In fact, the poles of $\Gamma(1-s)$ at $1 - s = -1, -3, -5, ...$, are cancelled by the seros of $\sin(\pi s/2)$ while the poles at $1-s = -2, -4, -6, ...$, are cancelled by the trivial zeros of $\zeta(1-s)$.

Further results on the Riemann zeta function follow from the functional equation but these rely on properties of the gamma function that we don't have time to consider.

Distribution of critical zeros

In the proof of the Prime Number Theorem we 'moved a line of integration' from $[c-iT, c+iT]$, with $c>1$, to $[1-\delta+iT,\, 1-\delta-iT]$ with some $\delta=\delta(T)>0$. We could, instead, move the line back to $[-R+iT,\, -R-iT]$ with arbitrarily large R, *independent* of T.

Again we apply Cauchy's Theorem with the contour $\mathcal{C}$, a rectangle with corners at $c-iT$, $c+iT$, $-R+iT$ and $-R-iT$. This time, though, the contour will contain poles. So

$$\frac{1}{2\pi i}\int_{\mathcal{C}} F(s)\,\frac{x^{s+1}ds}{s\,(s+1)} = \sum_{\text{poles in }\mathcal{C}} \operatorname{Res}\left(F(s)\,\frac{x^{s+1}}{s\,(s+1)}\right),$$

now a non-empty sum.

As deduced from the Functional Equation $\zeta(s)$ may have zeros in $0\le \operatorname{Re} s\le 1$, the ***critical strip***. There are zeros of $\zeta(s)$ to the left of $\sigma=0$ but there is no mystery to them, they are simple and lie at $s=-2n$, $n\ge 1$. Recall that a zero of $\zeta(s)$ becomes a simple pole of $\zeta'(s)/\zeta(s)$ and thus of $F(s)$, with residue $+1$.

Hence, assuming that $R>1$,

$$\sum_{\text{poles in }\mathcal{C}} \operatorname{Res}\left(F(s)\,\frac{x^{s+1}}{s\,(s+1)}\right) = F(0)\,x+F(-1)+\sum_{|\gamma|\le T}\frac{x^{\rho+1}}{\rho\,(\rho+1)} + \sum_{n\le R/2}\frac{x^{1-2n}}{2n\,(2n-1)}$$

Subtle point, the horizontal lines of $\mathcal{C}$ should not go through a critical zero of $\zeta(s)$ while the vertical line $\operatorname{Re} s=-R$ should not go through a trivial zero.

By choosing $R=T$ it can be shown that the first error found by truncating the original integral on the line $\operatorname{Re} s=c$ at $\pm T$ dominates all others and thus

$$\int_1^x \psi\,(t)\,dt = \frac{1}{2}x^2 - \sum_{|\gamma|\le T}\frac{x^{\rho+1}}{\rho\,(\rho+1)} + O\left(\frac{x\log^9 T}{T}\right). \qquad (4)$$

Definition 7.4 *Let*

$$N(T) = \left|\{\rho : \zeta(\rho)=0,\ 0<\operatorname{Re}\rho<1,\ 0<\operatorname{Im}\rho<T\}\right|.$$

The first result gives an upper bound on the number of critical zeros.

Lemma 7.5 *For $T > 0$ sufficiently large*

$$N(T+1) - N(T) \ll \log T. \tag{5}$$

Proof not given. ■

Note this does **not** actually say there are any non-trivial zeros satisfying $0 < \operatorname{Re}\rho < 1, T < \operatorname{Im}\rho < T+1$. It can, though, be shown though that for T sufficiently large we have

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O\left(\log^2 T\right).$$

This says quite accurately that there are many critical zeros.

In the sum over zeros in (4) we have $|x^\rho| = x^{\operatorname{Re}\rho}$. But we have already noted that the zeros of $\zeta(s)$ are symmetric around $\operatorname{Re} s = 1/2$, so half of them have $\operatorname{Re} s \geq 1/2$. Thus for such zeros $|x^\rho| \geq x^{1/2}$. In particular, the sum over zeros will be smallest if all zeros have real part equal to $1/2$, the Riemann Hypothesis.

Corollary 7.6 *On the Riemann Hypothesis*

$$\int_1^x \psi(t)\, dt = \frac{1}{2}x^2 + O\left(x^{3/2}\right). \tag{6}$$

Proof

$$\left|\sum_{|\gamma|\leq T} \frac{x^\rho}{\rho(\rho+1)}\right| \leq \sum_{|\gamma|\leq T} \frac{x^{\operatorname{Re}\rho}}{|\rho|\,|\rho+1|} = x^{1/2}\sum_{|\gamma|\leq T}\frac{1}{|\rho|\,|\rho+1|}.$$

I leave it to the student to use (5) to prove the sum over zeros converges. Thus the result follows on choosing $T = x$ say, in (4). ■

Unfortunately there is no efficient way to get a result on $\psi(t)$ from the integrated result in the Corollary.

Fortunately it is possible to prove an un-integrated version of (4):

Explicit formula Let $2 < T < x$. Then

$$\psi(x) = x - \sum_{|\operatorname{Im}\rho|\leq T} \frac{x^\rho}{\rho} + O\left(\frac{x\log^2 x}{T}\right). \tag{7}$$

Corollary 7.7 ***Prime Number Theorem with an error term.*** *On the Riemann Hypothesis*

$$\psi(x) = x + O\left(x^{1/2} \log^2 x\right).$$

Proof As noted above,

$$\left| \sum_{|\mathrm{Im}\,\rho| \le T} \frac{x^\rho}{\rho} \right| \le \sum_{|\mathrm{Im}\,\rho| \le T} \frac{x^{\mathrm{Re}\,\rho}}{|\rho|} = x^{1/2} \sum_{|\mathrm{Im}\,\rho| \le T} \frac{1}{|\rho|}.$$

For this sum, split $|\mathrm{Im}\,\rho| \le T$ into the union of $n \le |\mathrm{Im}\,\rho| < n+1$, for $n < T$. The first critical zero has imaginary part approximately 14.1347.. so we only need $n \ge 14$ in this sum. Thus

$$\begin{aligned}
\sum_{|\mathrm{Im}\,\rho| \le T} \frac{1}{|\rho|} &= \sum_{n=14}^{T} \sum_{n \le |\mathrm{Im}\,\rho| < n+1} \frac{1}{|\rho|} \le \sum_{n=14}^{T} \frac{1}{n} \sum_{n \le |\mathrm{Im}\,\rho| < n+1} 1 \\
&= \sum_{n=14}^{T} \frac{(N(n+1) - N(n))}{n} \\
&\ll \sum_{n=14}^{T} \frac{\log n}{n} \text{ using (5)} \\
&\ll \log T \sum_{n=14}^{T} \frac{1}{n} \ll \log^2 T.
\end{aligned}$$

Combining

$$\psi(x) = x + O\left(x^{1/2} \log^2 T\right) + O\left(\frac{x \log^2 x}{T}\right).$$

Simply choose T as a large power of x, i.e. x^{100}, to get the stated result. ■